

An analytically solvable model of probabilistic network dynamics

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We present a simple model of network dynamics that can be solved analytically for uniform networks. We obtain the dynamics of response of the system to perturbations. The analytical solution is an excellent approximation for random networks. A comparison with the scale-free network, though qualitatively similar, shows the effect of distinct topology.

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Recent advances in the understanding of complex social [1], biological [2], and technological [3] systems have revealed widespread if not universal properties of the topology of networks of association, interaction and communication. These properties, include small-world global connectivity [4], scale-free local connectivity distribution [5], and characteristic local motif structures [6]. Central to our understanding of complex systems [7] is characterizing their response to environmental stimuli. While much of the focus has been on robustness to random perturbation or directed attack, [8] the effectiveness of response requires satisfying a wider range of requirements including, for example, sensitivity to particular stimuli [9]. Indeed, one of the main functions of biological and social systems is the detection of specific stimuli that require collective (large scale) response in seeking desirable resources (foraging) or responding to dangers (“fight or flight”). Thus understanding the nature of system function and behavior from topological structure requires mapping the interaction structure given by a topological network onto the dynamics of system response [10]. Insofar as the network of interactions obtained in recent research characterizes the internal interactions of a system, these interactions must provide key information about the dynamics of response to external perturbations. Therefore, the construction of solvable dynamic models is essential for understanding the general features of the problem. In this Letter we propose such a model of probabilistic network dynamics, and we solve it analytically for uniform networks.

We consider a general network with N nodes. To each node i is assigned an *internal* state σ_i that can take the values 0 or 1. At each time step the state of a node is updated according to the following rule: either the state does not change, which happens with probability p or, with probability $(1 - p)$, it copies the state of one of its neighbors. This process describes, for example, the behavior of a group of high school students choosing to adopt one style of dress or another, or the propagation of a mutation through a species [11]. Since the states of a node are abstract labels, the change of one node to adopt the state of another can be considered a general model of influence propagation, with each node state a label for its own relevant physical property.

The 2^N states of the network can be labeled by a string of zeroes and ones describing the internal state of each node in sequence $(\sigma_{N-1}\sigma_{N-2}\dots\sigma_1\sigma_0)$. Alternatively, the states can be labeled by integers via $x = \sum_{j=0}^{N-1} \sigma_j 2^j$, with x varying between 0 and $2^N - 1$.

Let $P_t(x)$ be the probability of finding the network in the state x at time t and let the network evolve through asynchronous updates, where a single node is allowed to change at each time step. To find how this probability changes with time we define the auxiliary state \tilde{x}_k which is equal to x at all nodes except at node k , which has the opposite internal state. The probability of finding the network in the state x at time $t + 1$ can now be written as a sum of three terms: (a) the probability that the network was in state x at time t and that the selected node did not change plus (b) the probability that it was in the state x and the selected node copied the state of an identical neighbor plus (c) the probability that the network was in the state \tilde{x}_k at time t and that the node k was selected and its state $\tilde{\sigma}_k = 1 - \sigma_k$ changed to σ_k :

$$P_{t+1}(x) = pP_t(x) + \frac{(1-p)}{N} \sum_k \{P_t(x) \text{Prob}[\sigma_k \rightarrow \sigma_k] + P_t(\tilde{x}_k) \text{Prob}[\tilde{\sigma}_k \rightarrow \sigma_k]\}.$$

The probability $\text{Prob}[\sigma_k \rightarrow \sigma_k]$ is just the number of neighbors of node k in the state σ_k divided by the total number of neighbors (the degree) $d_k = \sum_{i=0}^{N-1} C_{ik}$, where C_{ik} is the connectivity (or adjacency) matrix. This can be written as

$$\frac{1}{d_k} \sum_{i=0}^{N-1} C_{ik} |1 - \sigma_i - \sigma_k|$$

The probability $\text{Prob}[\tilde{\sigma}_k \rightarrow \sigma_k]$ is also given by this formula, since $\tilde{\sigma}_k = 1 - \sigma_k$ and $\tilde{\sigma}_i = \sigma_i$ for $i \neq k$. Using these relations, we obtain the following master equation for the network dynamics:

$$P_{t+1}(x) = pP_t(x) + \frac{(1-p)}{N} \sum_{k=0}^{N-1} \frac{1}{d_k} \sum_{i=0}^{N-1} C_{ik} \times [1 - \sigma_i - \sigma_k] [P_t(x) + P_t(\tilde{x}_k)]. \quad (1)$$

Finding $P_t(x)$ for networks with arbitrary topologies can be very difficult. However, the problem can be

completely solved for fully connected networks, where $d_k = N - 1$. In this case the nodes are indistinguishable from each other and the states of the network can be labeled simply by counting the number of nodes in the internal state 1, given by $n(x) = \sum_i \sigma_i$. The probability of finding the network in the state labeled by n is related to $P(x)$ by

$$P(n(x)) = P(x) B(N, n) \quad (2)$$

where $B(N, n) = N!/[n!(N-n)!]$ is a binomial coefficient. We now simplify the last two terms on the right of Eq.(1). To do this we separate the sum over k into the cases $\sigma_k = 1$ and $\sigma_k = 0$. For the first of these terms we obtain

$$\begin{aligned} & \frac{1-p}{N(N-1)} \sum_k \sum_i [C_{ik} \sigma_i + C_{ik} (1 - \sigma_i)] P_t(x) \\ &= \frac{1-p}{N(N-1)} [n(n-1) + (N-n)(N-n-1)] P_t(x). \end{aligned}$$

For the third term we observe that $P_t(\tilde{x}_k)$ corresponds to the state $n-1$ if $\sigma_k = 1$ and to $n+1$ if $\sigma_k = 0$. When we separate the sum over k into the cases $\sigma_k = 1$ and $\sigma_k = 0$ we write $P_t(\tilde{x}_k; 1)$ and $P_t(\tilde{x}_k; 0)$ respectively. We obtain

$$\frac{(1-p)}{N(N-1)} [n(n-1)P_t(\tilde{x}_k; 1) + (N-n)(N-n-1)P_t(\tilde{x}_k; 0)].$$

Substituting these terms into the master equation and multiplying both sides by $B(N, n)$ we obtain, after some simplification,

$$\begin{aligned} P_{t+1}(n) &= pP_t(n) + \frac{1-p}{N(N-1)} \times \\ &\quad \{[n(n-1) + (N-n)(N-n-1)]P_t(n) \\ &\quad + (N-n+1)(n-1)P_t(n-1) \\ &\quad + (N-n-1)(n+1)P_t(n+1)\} \end{aligned} \quad (3)$$

For *uniform networks* where $d_k = d_0$ is the same for all nodes, if $d_0 < N - 1$ states with the same $n(x)$ can be distinguished by the way the internal states with $\sigma_k = 1$ are distributed among those with $\sigma_k = 0$. However, if we combine states with the same $n(x)$, the procedure described above can still be applied. In this case the factor d_k in the denominator is replaced by d_0 . However, on the average (with respect to the different states labeled by n), the counting of the number of neighbors must be multiplied by $d_0/(N-1)$, so that d_0 cancels and we get $(N-1)$ back in the denominator. Therefore equation (3) holds in this case as well. For random networks the degree of each node is nearly constant, and we can still use equation (3) as an approximation for the dynamics.

We now proceed to the calculation of the transition probabilities. The probabilities $P_t(n)$ define a vector P_t of $N+1$ components. The master equation (3) can be written in matrix form as $P_{t+1} = UP_t$ where the *evolution matrix* U is tridiagonal. The propagation of an initial probability vector requires the calculation of powers of U . Alternatively, we can diagonalize U and use its eigenvectors as a basis. This approach has been used [12] to calculate the eigenvalues of the transition matrix for

certain population models. Here we shall calculate not only the eigenvalues but also the eigenvectors, obtaining the complete solution of the dynamical problem.

The eigenvalues of U can be calculated for small matrices and extrapolated to matrices of arbitrary size. They are given by

$$\lambda_r = 1 - \frac{1-p}{N(N-1)} r(r-1).$$

with $r = 0, 1, \dots, N$. The only degeneracy occurs for $\lambda_0 = \lambda_1 = 1$. The other eigenvalues are all smaller than 1 and decrease towards $\lambda_N = p$.

Since U is not symmetric, its eigenvectors do not form an orthogonal set. Let $|a_r\rangle$ and $\langle b_r|$ be the right and left eigenvectors of U , with components a_{rm} and b_{rm} . Then

$$\sum_{r=0}^N \frac{1}{\Gamma_r} |a_r\rangle \langle b_r| = 1.$$

where $\langle b_{r'}|a_r\rangle = \Gamma_r \delta_{rr'}$ and $\Gamma_r = \sum_m a_{rm} b_{rm}$.

An initial vector $|v(0)\rangle$ containing the information about the probability of the different states at time zero can be projected using this resolution of unity and easily evolved:

$$|v(t)\rangle = U^t |v(0)\rangle = \sum_{r=0}^N \frac{1}{\Gamma_r} \langle b_r|v(0)\rangle \lambda_r^t |a_r\rangle.$$

The transition probability between two network states with $n = M$ and $n = L$ after a time t can now be calculated by taking the components of the initial vector as $v_m(0) = \delta_{M,m}$ and projecting the evolved state onto the state with components $\delta_{L,m}$:

$$P(L, t; M, 0) = \sum_{r=0}^N \frac{1}{\Gamma_r} b_{rM} a_{rL} \lambda_r^t.$$

The coefficients a_{rm} follow a recursion relation that can be derived directly from the eigenvalue equation for U . For $r = 0$ and $r = 1$ the eigenvectors can be found immediately:

$$|a_0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad |a_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} \quad (4)$$

and

$$\begin{aligned} |b_0\rangle &= (1 \ 1 \ 1 \ \dots \ 1 \ 1), \\ |b_1\rangle &= (N \ N-2 \ N-4 \ \dots \ -N+2 \ -N). \end{aligned} \quad (5)$$

In order to calculate the remaining eigenvectors, we define the auxiliary eigenvalues μ_r by

$$\mu_r = (1 - \lambda_r) \frac{N(N-1)}{1-p} = r(r-1) \quad (6)$$

and the auxiliary coefficients

$$A_{rm} = m(N-m)a_{rm} \quad m = 1, 2, \dots, N-1. \quad (7)$$

The recursion relation for the A_{rm} can be written explicitly as:

$$A_{r\,m+1} - 2A_{rm} + A_{r\,m-1} = -\frac{\mu_r}{N} \left(\frac{A_{rm}}{m} + \frac{A_{rm}}{N-m} \right). \quad (8)$$

A generating function is now defined as

$$f_r(x) = \sum_{m=1}^{N-1} A_{rm} x^m$$

(note that $A_{r0} = A_{rN} = 0$). Multiplying eq.(8) by x^m and summing over m we get, on the left side,

$$\frac{f_r}{x} (1-x)^2 - A_{r1} + A_{r\,N-1} x^N.$$

In order to write down the right side of eq.(8) we define the auxiliary functions

$$g_r(x) = \sum_{m=1}^{N-1} \frac{A_{rm}}{m} x^m, \quad h_r(x) = \sum_{m=1}^{N-1} \frac{A_{rm}}{N-m} x^m. \quad (9)$$

It is easy to check that $dg_r/dx = f_r/x$ and $dh_r/dx = Nh_r/x - f_r/x$. After multiplying Eq.(8) by x^m and summing over m , we differentiate both sides with respect to x to obtain

$$\frac{d}{dx} \left[\frac{(1-x)^2}{x} f_r(x) - A_{r\,N-1} x^N \right] = -\frac{\mu_r}{x} h_r(x). \quad (10)$$

The solution of the differential equation for h_r can be obtained in terms of its Green function, satisfying $dG/dx - NG/x = \delta(x-y)$. In this case G is given by $(x/y)^N$ if $x > y$ and zero otherwise. Therefore,

$$h_r(x) = x^N \left(\alpha - \int_{-\infty}^x \frac{f_r(y)}{y^{N+1}} dy \right). \quad (11)$$

Substituting Eq.(11) into (10), re-arranging the terms and differentiating once again with respect to x , we obtain

$$\frac{d}{dx} \left[\frac{1}{x^{N-1}} \frac{d}{dx} \left(\frac{(1-x)^2}{x} f_r(x) \right) \right] = \mu_r \frac{f_r(x)}{x^{N+1}}.$$

Finally, defining

$$F_r(x) = \frac{(1-x)^2}{x} f_r(x) \quad (12)$$

we obtain the differential equation

$$F_r'' - \frac{N-1}{x} F_r' - \frac{\mu_r}{x} \frac{F_r}{(1-x)^2} = 0. \quad (13)$$

Letting $\phi_r(x) = \sum a_{rm} x^m$ then

$$\phi_r(x) = \sum \frac{A_{rm}}{m(N-m)} x^m = \frac{1}{N} (g_r + h_r).$$

Differentiating with respect to x , using Eq.(11), dividing by x^{N-1} and differentiating again, we find

$$\phi_r'' - \frac{N-1}{x} \phi_r' - \frac{1}{x} \frac{F_r}{(1-x)^2} = 0.$$

Comparing with Eq.(13) we see that $\phi_r = -F_r/\mu_r$. Therefore, except for a normalization, *the generating function for the coefficients a_{rm} , $\phi_r(x)$, is equal to $F_r(x)$.*

For $r = 0$ or $r = 1$, $\mu_r = 0$ and the two independent solutions of eq.(13) are $F_0(x) = 1 + x^N$ and $F_1(x) = 1 - x^N$, which correspond to the two degenerate eigenvectors $|a_0\rangle$ and $|a_1\rangle$. For $r = 2$ and $r = 3$ the solution can also be found explicitly; the general formula can then be extrapolated from these simple cases. We find

$$F_r(x) = (1-x)^{1-r} \left[1 + \sum_{p=1}^{r-1} d_{rp} x^p \right]. \quad (14)$$

with

$$d_{rp} = (-1)^p \frac{B(r-1, p) B(N+r-1, p)}{B(N-1, p)}.$$

Finally, the coefficients of the r -th eigenvector are given by

$$a_{rm} = \frac{1}{m!} \left. \frac{d^m F_r(x)}{dx^m} \right|_{x=0}.$$

Since F_r is the product of two simple functions, its derivative can be calculated explicitly at $x = 0$. Writing $F_r(x) = N_r(x) Q_r(x)$ with

$$Q_r(x) = (1-x)^{1-r} \quad \text{and} \quad N_r(x) = 1 + \sum d_{rp} x^p$$

we find

$$a_{rm} = \frac{1}{m!} \sum_{p=0}^m B(m, p) \left. \frac{d^{m-p} N_r}{dx^{m-p}} \right|_{(x=0)} \left. \frac{d^p Q_r}{dx^p} \right|_{(x=0)}.$$

Working out the derivatives we find the explicit formula valid for $r \geq 2$.

$$a_{rm} = \sum_{p=0}^{r-1} B(m-p+r-2, r-2) d_{rp} \quad (15)$$

for $m = 1, 2, \dots, N-1$, with $a_{r0} = 1$ and $a_{rN} = (-1)^r$.

From the recursion relations we find that the coefficients of the left eigenvectors are given by

$$b_{rm} = a_{rm} [m(N-m)/N] \quad (16)$$

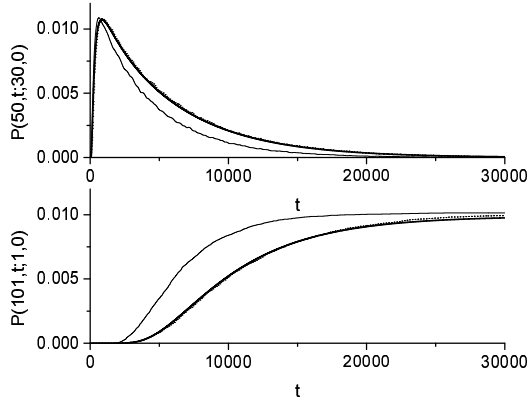


FIG. 1: Transition probabilities for a network with $N = 101$ and $p = 0.1$. The lines correspond to our theoretical calculation (thick) and to simulations for random (dotted) and scale-free (thin) networks, both with an average of 6 connections per node. The numerical probabilities were computed running the simulations 2×10^5 times. The dotted line is nearly indistinguishable from the thick line.

for $m = 1, 2, \dots, N-1$, with $b_{r0} = b_{rN} = 0$. Finally, the normalization factors Γ_r can also be obtained explicitly:

$$\Gamma_r = \frac{r! B(N+r-1, r)}{(2r-1) B(N, r)} \quad (17)$$

for $r = 2, \dots, N$ and $\Gamma_0 = 2$, $\Gamma_1 = 2N$.

We are finally in a position to state some important results concerning the dynamics. The transition probability of starting the network at time zero with $n = M$ and finding it at a later time t with $n = L$ can be computed using Eqs.(4), (5), (15), (16) and (17):

$$P(L, t; M, 0) = \left[\frac{N-M}{N} - \frac{3M(N-M)}{(N+1)N} \lambda_2^t \right] \delta_{L0} + \left[\frac{M}{N} - \frac{3M(N-M)}{(N+1)N} \lambda_2^t \right] \delta_{LN} + \frac{6M(N-M)}{N(N^2-1)} (1-\delta_{L0})(1-\delta_{LN}) \lambda_2^t + \sum_{r=3}^N \frac{1}{\Gamma_r} b_{rM} a_{rL} \lambda_r^t.$$

Since all eigenvalues (except for λ_0 and λ_1) are smaller than one, in the limit of long times the transition probability is dominated by $\lambda_0 = \lambda_1 = 1$ and by the largest non-trivial eigenvalue, λ_2 , whose contributions we have written down explicitly. For large networks we can approximate $\lambda_2^t \approx \exp[-2(1-p)t/N^2]$, so that the characteristic duration of the transition process is $\tau = N^2/(1-p)$, which increases with the square of the network size.

The only two possible asymptotic states are $n = 0$ and $n = N$, whose transition probabilities from an initial state $n = M$ are $(N-M)/N$ and M/N respectively. A typical transition probability $P(L, t; M, 0)$ for $L, M \neq 0, N$ starts at zero if $L \neq M$, reaches a maximum and decreases back to zero. This represents the probability that a perturbation initially affecting M nodes will lead to a response by L nodes at a time t later. Figure 1 shows an example for a network with $N = 101$ nodes and $p = 0.1$. The estimate $\tau \approx 11000$ works well for all the transition probabilities shown. These results show that the theoretical model for fully connected networks is an excellent approximation for the average behavior of sparsely connected random networks. The theory also reproduces qualitatively the behavior of scale-free networks. The deviations from the theory in this case reflect the significant topological differences between the two networks. As a final remark we note that the case of more internal states per node can in principle be treated in a similar fashion. However, although Eq.(1) can be easily generalized, Eq.(3) would be more complicated, since it would take more than a single integer $n(x)$ to describe the network state.

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